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Penyelesaian Persamaan Poisson 2D dengan Menggunakan Metode *Gauss-Seidel* dan *Conjugate Gradient*

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Abstract— *In this paper we focus on solution of 2D Poisson equation numerically. 2D Poisson equation is a partial differential equation of second order elliptical type. This equation is a particular form or non-homogeneous form of the Laplace equation. The solution of 2D Poisson equation is performed numerically using Gauss Seidel method and Conjugate Gradient method. The result is the value using Gauss Seidel method and Conjugate Gradient method is same. But, consider the iteration process, the convergence of the value is reached faster using Conjugate Gradient method.*

Keywords—2D Poisson equation, Gauss-Seidel method, and Conjugate Gradient method.

1. PENDAHULUAN

Persamaan Poisson adalah salah satu persamaan diferensial parsial eliptik dua dimensi. Seperti kita ketahui bahwa persamaan diferensial parsial memegang peranan penting di dalam penggambaran fisis, dimana besaran-besaran didalamnya berubah terhadap ruang dan waktu. Penggunaan persamaan diferensial parsial akan ditemukan pada penyelesaian masalah-masalah fenomena alam. Penggunaan persamaan diferensial tidak terbatas pada masalah fisika saja, tetapi lebih luas lagi dalam bidang sains dan teknologi.

Masalah-masalah tersebut dalam kenyataannya sulit untuk dipecahkan dengan cara analitik biasa, sehingga metode numerik perlu diterapkan untuk menyelesaikannya. Hal ini dapat memudahkan para peneliti dalam melakukan simulasi dalam suatu permasalahan.

Metode penyelesaian numerik untuk persamaan diferensial eliptik diklasifikasikan dalam dua kategori, yaitu metode beda hingga (*finite difference methods*) dan elemen hingga (*finite element methods*). Penyelesaian permasalahan persamaan Poisson secara numerik dapat dilakukan dengan metode apa saja asalkan sistem persamaan linier yang bersesuaian dengan skema beda memenuhi syarat-syarat dalam metode yang dipakai.

2. TINJAUAN PUSTAKA

2.1 Persamaan Poisson 2D

Metode penyelesaian numerik untuk persamaan diferensial eliptik pada penelitian ini adalah dengan menggunakan metode beda hingga (*finite difference methods*).

Pendekatan beda hingga di dimensi dua dari persamaan Poisson dua dimensi

$$-\nabla^2 u(x,y) = f(x,y)$$

yang ekivalen dengan

$$-u_{xx} - u_{yy} = f(x,y)$$

dan juga

$$-\frac{\partial^2 u(x,y)}{\partial x^2} - \frac{\partial^2 u(x,y)}{\partial y^2} = f(x,y). \tag{2.1}$$

Dimana $f(x,y)$ adalah fungsi yang ditentukan, adalah sebagai berikut. Pendekatan beda hingga menggunakan asumsi bahwa fungsi $u(x)$ adalah kontinu sedemikian hingga nilai-nilai fungsi di sekitar $x = x_j$ beserta turunannya dapat didekati dengan ekspansi deret Taylor. Pendekatan turunan kedua dengan beda pusat dari $u(x)$ adalah

$$u_{xx}(x_j) = \left.\frac{d^2u(x)}{dx^2}\right|_{x=x_j} = \frac{u(x_j+h) - 2u(x_j) + u(x_j-h)}{h^2} + O(h^2)$$

atau ekivalen dengan

$$\frac{\partial^2 u}{\partial x^2}(x_j, y_k) = \frac{u_{j+1,k} - 2u_{j,k} + u_{j-1,k}}{h^2} + O(h^2)$$

$$\frac{\partial^2 u}{\partial y^2}(x_j, y_k) = \frac{u_{j,k+1} - 2u_{j,k} + u_{j,k-1}}{k^2} + O(k^2)$$

[3].

2.2 Penurunan Skema pada persamaan Poisson

Pandang persamaan Poisson (2.1) dan dengan syarat batas

$$\alpha u + \beta u_{\mathbf{n}} = \gamma, \quad (x,y) \in \partial\Omega.$$

Dalam permasalahan ini dipilih syarat batas tipe Dirichlet

$$u = \gamma \tag{2.2}$$

yaitu ketika $\alpha = 1$ dan $\beta = 0$, dan domain

$$\Omega = \{(x,y) | 0 \le x \le a, 0 \le y \le b\} \tag{2.3}$$

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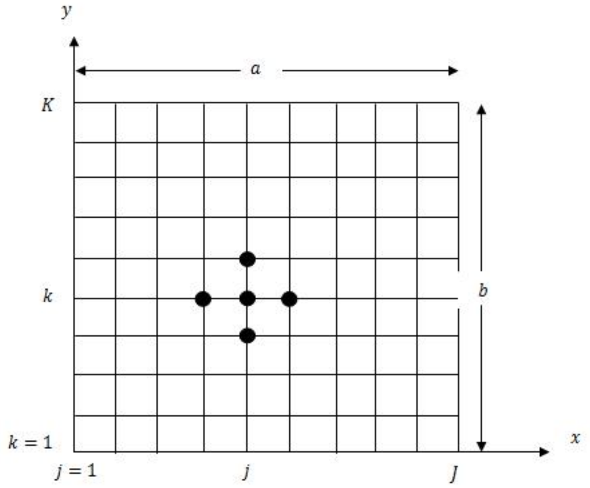
Stensil untuk solusi persamaan Poisson adalah:



Gambar 1 Stensil untuk persamaan Poisson

Persamaan (2.1) akan didekati dengan beda pusat, dengan memisalkan $U_{j,k}$ dinotasikan sebagai pendekatan beda hingga dari $u(j\Delta x, k\Delta y)$.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\Delta x}\left(\frac{U_{j+1,k} - U_{j,k}}{\Delta x} - \frac{U_{j,k} - U_{j-1,k}}{\Delta x}\right) = \frac{U_{j+1,k} - 2U_{j,k} + U_{j-1,k}}{\Delta x^2} \tag{2.4}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\Delta y}\left(\frac{U_{j,k+1} - U_{j,k}}{\Delta y} - \frac{U_{j,k} - U_{j,k-1}}{\Delta y}\right) = \frac{U_{j,k+1} - 2U_{j,k} + U_{j,k-1}}{\Delta y^2} \tag{2.5}$$

Substitusi persamaan (2.4) dan (2.5) ke persamaan (2.1), sebagai berikut

$$-\frac{U_{j+1,k} - 2U_{j,k} + U_{j-1,k}}{\Delta x^2} - \frac{U_{j,k+1} - 2U_{j,k} + U_{j,k-1}}{\Delta y^2} = f_{j,k}$$

$$\frac{-U_{j+1,k} + 2U_{j,k} - U_{j-1,k}}{\Delta x^2} + \frac{-U_{j,k+1} + 2U_{j,k} - U_{j,k-1}}{\Delta y^2} = f_{j,k}$$

$$\frac{\Delta y^2(-U_{j+1,k} + 2U_{j,k} - U_{j-1,k}) + \Delta x^2(-U_{j,k+1} + 2U_{j,k} - U_{j,k-1})}{\Delta x^2 \Delta y^2} = f_{j,k}$$

$$-\Delta y^2 U_{j+1,k} + 2\Delta y^2 U_{j,k} - \Delta y^2 U_{j-1,k} - \Delta x^2 U_{j,k+1} + 2\Delta x^2 U_{j,k} - \Delta x^2 U_{j,k-1} = \Delta x^2 \Delta y^2 f_{j,k}$$

$$-\Delta y^2 U_{j+1,k} - \Delta y^2 U_{j-1,k} - \Delta x^2 U_{j,k+1} - \Delta x^2 U_{j,k-1} + 2(\Delta x^2 + \Delta y^2)U_{j,k} = \Delta x^2 \Delta y^2 f_{j,k}$$

$$U_{j,k} - \frac{\Delta y^2}{2(\Delta x^2 + \Delta y^2)}U_{j+1,k} - \frac{\Delta y^2}{2(\Delta x^2 + \Delta y^2)}U_{j-1,k} - \frac{\Delta x^2}{2(\Delta x^2 + \Delta y^2)}U_{j,k+1} - \frac{\Delta x^2}{2(\Delta x^2 + \Delta y^2)}U_{j,k-1} = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}f_{j,k}$$

$$U_{j,k} - \frac{\Delta y^2}{2(\Delta x^2 + \Delta y^2)}(U_{j+1,k} + U_{j-1,k}) - \frac{\Delta x^2}{2(\Delta x^2 + \Delta y^2)}(U_{j,k+1} + U_{j,k-1}) = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}f_{j,k}$$

$$U_{j,k} - \theta_x(U_{j+1,k} + U_{j-1,k}) - \theta_y(U_{j,k+1} + U_{j,k-1}) = \theta_{xy} f_{j,k} \tag{2.6}$$

dimana

$$\theta_x = \frac{\Delta y^2}{2(\Delta x^2 + \Delta y^2)}, \theta_y = \frac{\Delta x^2}{2(\Delta x^2 + \Delta y^2)}, \text{dan } \theta_{xy} = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}. \tag{2.7}$$

Dengan menggunakan persamaan (2.6) dapat dicari pendiskritan sebagai berikut:

Untuk $J = K = 4$ pada (titik *j*, titik *k*)

$$(2,2) \Rightarrow U_{2,2} - \theta_x(U_{3,2} + U_{1,2}) - \theta_y(U_{2,3} + U_{2,1}) = \theta_{xy} f_{2,} \quad (2.8)$$
$$\Rightarrow U_{2,2} - \theta_x U_{3,2} - \theta_y U_{2,3} = \theta_{xy} f_{2,2} + \theta_x \gamma_{1,2} + \theta_y \gamma_{2,1}$$
$$(3,2) \Rightarrow U_{3,2} - \theta_x(U_{4,2} + U_{2,2}) - \theta_y(U_{3,3} + U_{3,1}) = \theta_{xy} f_{3,2} \quad (2.9)$$
$$\Rightarrow -\theta_x U_{2,2} + U_{3,2} - \theta_y U_{3,3} = \theta_{xy} f_{3,2} + \theta_x \gamma_{4,2} + \theta_y \gamma_{3,1}$$
$$(2,3) \Rightarrow U_{2,3} - \theta_x(U_{3,3} + U_{1,3}) - \theta_y(U_{2,4} + U_{2,2}) = \theta_{xy} f_{2,3} \quad (2.10)$$
$$\Rightarrow -\theta_y U_{2,2} + U_{2,3} - \theta_x U_{3,3} = \theta_{xy} f_{2,3} + \theta_x \gamma_{1,3} + \theta_y \gamma_{2,4}$$
$$(3,3) \Rightarrow U_{3,3} - \theta_x(U_{4,3} + U_{2,3}) - \theta_y(U_{3,4} + U_{3,2}) = \theta_{xy} f_{3} \quad (2.11)$$
$$\Rightarrow -\theta_y U_{3,2} - \theta_x U_{2,3} + U_{3,3} = \theta_{xy} f_{3,3} + \theta_x \gamma_{4,3} + \theta_y \gamma_{3,4}$$

Dari persamaan (2.8) sampai (2.11), dapat dibentuk suatu sistem persamaan linier yang bersesuaian sebagai berikut:

$$\begin{bmatrix} 1 & -\theta_x & -\theta_y & 0 \\ -\theta_x & 1 & 0 & -\theta_y \\ -\theta_y & 0 & 1 & -\theta_x \\ 0 & -\theta_y & -\theta_x & 1 \end{bmatrix} \begin{bmatrix} U_{2,2} \\ U_{3,2} \\ U_{2,3} \\ U_{3,3} \end{bmatrix} = \begin{bmatrix} \theta_{xy} f_{2,2} + \theta_x \gamma_{1,2} + \theta_y \gamma_{2,1} \\ \theta_{xy} f_{3,2} + \theta_x \gamma_{4,2} + \theta_y \gamma_{3,1} \\ \theta_{xy} f_{2,3} + \theta_x \gamma_{1,3} + \theta_y \gamma_{2,4} \\ \theta_{xy} f_{3,3} + \theta_x \gamma_{4,3} + \theta_y \gamma_{3,4} \end{bmatrix}$$

Untuk $J = K = 5$ pada (titik *j*, titik *k*)

$$(2,2) \Rightarrow U_{2,2} - \theta_x U_{3,2} - \theta_y U_{2,3} = \theta_{xy} f_{2,2} + \theta_x \gamma_{1,2} + \theta_y \gamma_{2,1} \quad (2.12)$$
$$(3,2) \Rightarrow U_{3,2} - \theta_x U_{4,2} - \theta_x U_{2,2} - \theta_y U_{3,3} = \theta_{xy} f_{3,2} + \theta_y \gamma_{3,1} \quad (2.13)$$
$$(4,2) \Rightarrow U_{4,2} - \theta_x U_{3,2} - \theta_y U_{4,3} = \theta_{xy} f_{4,2} + \theta_x \gamma_{5,2} + \theta_y \gamma_{4,1} \quad (2.14)$$
$$(2,3) \Rightarrow U_{2,3} - \theta_x U_{3,3} - \theta_y U_{2,4} - \theta_y U_{2,2} = \theta_{xy} f_{2,3} + \theta_x \gamma_{1,3} \quad (2.15)$$
$$(3,3) \Rightarrow U_{3,3} - \theta_x U_{4,3} - \theta_x U_{2,3} - \theta_y U_{3,4} - \theta_y U_{3,2} = \theta_{xy} f_{3,3} \quad (2.16)$$
$$(4,3) \Rightarrow U_{4,3} - \theta_x U_{3,3} - \theta_y U_{4,4} - \theta_y U_{4,2} = \theta_{xy} f_{4,3} + \theta_x \gamma_{5,3} \quad (2.17)$$
$$(2,4) \Rightarrow U_{2,4} - \theta_x U_{3,4} - \theta_y U_{2,3} = \theta_{xy} f_{2,4} + \theta_x \gamma_{1,4} + \theta_y \gamma_{2,5} \quad (2.18)$$
$$(3,4) \Rightarrow U_{3,4} - \theta_x U_{4,4} - \theta_x U_{2,4} - \theta_y U_{3,3} = \theta_{xy} f_{3,4} + \theta_y \gamma_{3,5} \quad (2.19)$$
$$(4,4) \Rightarrow U_{4,4} - \theta_x U_{3,4} - \theta_y U_{4,3} = \theta_{xy} f_{4,4} + \theta_x \gamma_{5,4} + \theta_y \gamma_{4,5} \quad (2.20)$$

Dari persamaan (2.12) sampai dengan persamaan (2.20) dapat dibentuk suatu sistem persamaan linier yang bersesuaian sebagai berikut:

$$\begin{bmatrix} 1 & -\theta_x & 0 & -\theta_y & 0 & 0 & 0 & 0 & 0 \\ -\theta_x & 1 & -\theta_x & 0 & -\theta_y & 0 & 0 & 0 & 0 \\ 0 & -\theta_x & 1 & 0 & 0 & -\theta_y & 0 & 0 & 0 \\ -\theta_y & 0 & 0 & 1 & -\theta_x & 0 & -\theta_y & 0 & 0 \\ 0 & -\theta_y & 0 & -\theta_x & 1 & -\theta_x & 0 & -\theta_y & 0 \\ 0 & 0 & -\theta_y & 0 & -\theta_x & 1 & 0 & 0 & -\theta_y \\ 0 & 0 & 0 & -\theta_y & 0 & 0 & 1 & -\theta_x & 0 \\ 0 & 0 & 0 & 0 & -\theta_y & 0 & -\theta_x & 1 & -\theta_x \\ 0 & 0 & 0 & 0 & 0 & -\theta_y & 0 & -\theta_x & 1 \end{bmatrix} \begin{bmatrix} U_{2,2} \\ U_{3,2} \\ U_{4,2} \\ U_{2,3} \\ U_{3,3} \\ U_{4,3} \\ U_{2,4} \\ U_{3,4} \\ U_{4,4} \end{bmatrix} = \begin{bmatrix} \theta_{xy} f_{2,2} + \theta_x \gamma_{1,2} + \theta_y \gamma_{2,1} \\ \theta_{xy} f_{3,2} + \theta_y \gamma_{3,1} \\ \theta_{xy} f_{4,2} + \theta_x \gamma_{5,2} + \theta_y \gamma_{4,1} \\ \theta_{xy} f_{2,3} + \theta_x \gamma_{1,3} \\ \theta_{xy} f_{3,3} \\ \theta_{xy} f_{4,3} + \theta_x \gamma_{5,3} \\ \theta_{xy} f_{2,4} + \theta_x \gamma_{1,4} + \theta_y \gamma_{2,5} \\ \theta_{xy} f_{3,4} + \theta_y \gamma_{3,5} \\ \theta_{xy} f_{4,4} + \theta_x \gamma_{5,4} + \theta_y \gamma_{4,5} \end{bmatrix}$$

Untuk $J = K = 6$ pada (titik *j*, titik *k*)

$$(2,2) \Rightarrow U_{2,2} - \theta_x U_{3,2} - \theta_y U_{2,3} = \theta_{xy} f_{2,2} + \theta_x \gamma_{1,2} + \theta_y \gamma_{2,1} \quad (2.21)$$
$$(3,2) \Rightarrow U_{3,2} - \theta_x U_{4,2} - \theta_x U_{2,2} - \theta_y U_{3,3} = \theta_{xy} f_{3,2} + \theta_y \gamma_{3,1} \quad (2.22)$$
$$(4,2) \Rightarrow U_{4,2} - \theta_x U_{5,2} - \theta_x U_{3,2} - \theta_y U_{4,3} = \theta_{xy} f_{4,2} + \theta_y \gamma_{4,1} \quad (2.23)$$
$$(5,2) \Rightarrow U_{5,2} - \theta_x U_{4,2} - \theta_y U_{5,3} = \theta_{xy} f_{5,2} + \theta_x \gamma_{6,2} + \theta_y \gamma_{5,1} \quad (2.24)$$
$$(2,3) \Rightarrow U_{2,3} - \theta_x U_{3,3} - \theta_y U_{2,4} - \theta_y U_{2,2} = \theta_{xy} f_{2,3} + \theta_x \gamma_{1,3} \quad (2.25)$$
$$(3,3) \Rightarrow U_{3,3} - \theta_x U_{4,3} - \theta_x U_{2,3} - \theta_y U_{3,4} - \theta_y U_{3,2} = \theta_{xy} f_{3,3} \quad (2.26)$$
$$(4,3) \Rightarrow U_{4,3} - \theta_x U_{5,3} - \theta_x U_{3,3} - \theta_y U_{4,4} - \theta_y U_{4,2} = \theta_{xy} f_{4,3} \quad (2.27)$$
$$(5,3) \Rightarrow U_{5,3} - \theta_x U_{4,3} - \theta_y U_{5,4} - \theta_y U_{5,2} = \theta_{xy} f_{5,3} + \theta_x \gamma_{6,3} \quad (2.28)$$
$$(2,4) \Rightarrow U_{2,4} - \theta_x U_{3,4} - \theta_y U_{2,5} - \theta_y U_{2,3} = \theta_{xy} f_{2,4} + \theta_x \gamma_{1,4} \quad (2.29)$$
$$(3,4) \Rightarrow U_{3,4} - \theta_x U_{4,4} - \theta_x U_{2,4} - \theta_y U_{3,5} - \theta_y \gamma_{3,3} = \theta_{xy} f_{3,4} \quad (2.30)$$
$$(4,4) \Rightarrow U_{4,4} - \theta_x U_{5,4} - \theta_x U_{3,4} - \theta_y U_{4,5} - \theta_y U_{4,3} = \theta_{xy} f_{4,4} \quad (2.31)$$
$$(5,4) \Rightarrow U_{5,4} - \theta_x U_{4,4} - \theta_y U_{5,5} - \theta_y U_{5,3} = \theta_{xy} f_{5,4} + \theta_x U_{6,4} \quad (2.32)$$
$$(2,5) \Rightarrow U_{2,5} - \theta_x U_{3,5} - \theta_y U_{2,4} = \theta_{xy} f_{2,5} + \theta_x \gamma_{1,5} + +\theta_y \gamma_{2} \quad (2.33)$$
$$(3,5) \Rightarrow U_{3,5} - \theta_x U_{4,5} - \theta_x U_{2,5} - \theta_y U_{3,4} = \theta_{xy} f_{3,5} - \theta_y \gamma_{3,} \quad (2.34)$$
$$(4,5) \Rightarrow U_{4,5} - \theta_x U_{5,5} - \theta_x U_{3,5} - \theta_y U_{4,4} = \theta_{xy} f_{4,5} - \theta_y U_{4,} \quad (2.35)$$
$$(5,5) \Rightarrow U_{5,5} - \theta_x U_{4,5} - \theta_y U_{5,4} = \theta_{xy} f_{5,5} + \theta_x U_{6,5} - \theta_y U_{5,} \quad (2.36)$$

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Dari persamaan (2.21) sampai dengan persamaan (2.36) dapat dibentuk suatu sistem persamaan linier yang bersesuaian sebagai berikut:

$$\begin{bmatrix}
1 & -\theta_x & 0 & 0 & -\theta_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\theta_x & 1 & -\theta_x & 0 & 0 & -\theta_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\theta_x & 1 & -\theta_x & 0 & 0 & -\theta_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\theta_x & 1 & 0 & 0 & 0 & -\theta_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\theta_y & 0 & 0 & 0 & 1 & -\theta_x & 0 & 0 & -\theta_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\theta_y & 0 & 0 & -\theta_x & 1 & -\theta_x & 0 & 0 & -\theta_y & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\theta_y & 0 & 0 & -\theta_x & 1 & -\theta_x & 0 & 0 & -\theta_y & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\theta_y & 0 & 0 & -\theta_x & 1 & 0 & 0 & 0 & -\theta_y & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\theta_y & 0 & 0 & 0 & 1 & -\theta_x & 0 & 0 & -\theta_y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\theta_y & 0 & 0 & -\theta_x & 1 & -\theta_x & 0 & 0 & -\theta_y & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\theta_y & 0 & 0 & -\theta_x & 1 & -\theta_x & 0 & 0 & -\theta_y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\theta_y & 0 & 0 & -\theta_x & 1 & 0 & 0 & 0 & -\theta_y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\theta_y & 0 & 0 & 0 & 1 & -\theta_x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\theta_y & 0 & 0 & -\theta_x & 1 & -\theta_x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\theta_y & 0 & 0 & -\theta_x & 1 & -\theta_x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\theta_y & 0 & 0 & -\theta_x & 1
\end{bmatrix}
\begin{bmatrix} U_{2,2} \\ U_{3,2} \\ U_{4,2} \\ U_{5,2} \\ U_{2,3} \\ U_{3,3} \\ U_{4,3} \\ U_{5,3} \\ U_{2,4} \\ U_{3,4} \\ U_{4,4} \\ U_{5,4} \\ U_{2,5} \\ U_{3,5} \\ U_{4,5} \\ U_{5,5} \end{bmatrix}
=
\begin{bmatrix}
\theta_{xy} f_{2,2} + \theta_x \gamma_{1,2} + \theta_y \gamma_{2,1} \\
\theta_{xy} f_{3,2} + \theta_y \gamma_{3,1} \\
\theta_{xy} f_{4,2} + \theta_y \gamma_{4,1} \\
\theta_{xy} f_{5,2} + \theta_x \gamma_{6,2} + \theta_y \gamma_{5,1} \\
\theta_{xy} f_{2,3} + \theta_x \gamma_{1,3} \\
\theta_{xy} f_{3,3} \\
\theta_{xy} f_{4,3} \\
\theta_{xy} f_{5,3} + \theta_x \gamma_{6,3} \\
\theta_{xy} f_{2,4} + \theta_x \gamma_{1,4} \\
\theta_{xy} f_{3,4} \\
\theta_{xy} f_{4,4} \\
\theta_{xy} f_{5,4} \\
\theta_{xy} f_{2,5} + \theta_x \gamma_{1,5} + \theta_y \gamma_{2,6} \\
\theta_{xy} f_{3,5} + \theta_y \gamma_{3,6} \\
\theta_{xy} f_{4,5} + \theta_y \gamma_{4,6} \\
\theta_{xy} f_{5,5} + \theta_x \gamma_{6,5} + \theta_y \gamma_{5,6}
\end{bmatrix}$$

Dengan memisalkan

$$\mathbf{U}_K = \left[U_{2,k}, U_{3,k}, \dots, U_{J-1,k}\right]^T$$

dan

$$\mathbf{U} = \left[\mathbf{U}_2^T, \mathbf{U}_3^T, \dots, \mathbf{U}_{K-1}^T\right]^T$$

maka bentuk sistem persamaan linier secara umum adalah:

$$\mathbf{AU} = \mathbf{b}$$

$$\begin{bmatrix} \mathbf{C}_2 & \mathbf{D}_2 & & \\ \mathbf{D}_2 & \mathbf{C}_3 & \mathbf{D}_3 & \\ & & \ddots & \\ & & \mathbf{D}_{K-1} & \mathbf{C}_{K-1} \end{bmatrix} \begin{bmatrix} \mathbf{U}_2 \\ \mathbf{U}_3 \\ \vdots \\ \mathbf{U}_{K-1} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_2 \\ \mathbf{b}_3 \\ \vdots \\ \mathbf{b}_{K-1} \end{bmatrix}$$

dengan

$$\mathbf{C}_k = \begin{bmatrix} 1 & -\theta_x & & \\ -\theta_x & 1 & -\theta_x & \\ & & \ddots & \\ & & -\theta_x & 1 \end{bmatrix}, \quad \mathbf{D}_k = \begin{bmatrix} -\theta_y & & & \\ & -\theta_y & & \\ & & \ddots & \\ & & & -\theta_y \end{bmatrix}$$

dan

$$\boldsymbol{b}_k = \theta_{xy} \begin{bmatrix} f_{2,k} \\ f_{3,k} \\ \vdots \\ f_{J-1,k} \end{bmatrix} + \theta_x \begin{bmatrix} \gamma_{1,k} \\ 0 \\ \vdots \\ 0 \\ \gamma_{J,k} \end{bmatrix} + \theta_y \begin{bmatrix} \gamma_{2,1} \\ \gamma_{3,1} \\ \vdots \\ 0 \\ \gamma_{J-1,1} \end{bmatrix} + \theta_y \begin{bmatrix} \gamma_{2,K} \\ \gamma_{3,K} \\ \vdots \\ 0 \\ \gamma_{J-1,K} \end{bmatrix}$$

Matriks $\mathbf{C}_k$ adalah matriks tridiagonal dan $\mathbf{D}_k$ adalah matriks diagonal [1].

2.3 Kesalahan Pemotongan

Skema beda:

$$U_{j,k} - \frac{\Delta y^2}{2(\Delta x^2 + \Delta y^2)}\left(U_{j+1,k} + U_{j-1,k}\right) - \frac{\Delta x^2}{2(\Delta x^2 + \Delta y^2)}\left(U_{j,k+1} + U_{j,k-1}\right) = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)} f_{j,k} \quad (2.37)$$

Cara untuk mencari kesalahan pemotongan adalah dengan ekspansi deret Taylor.

Ekspansi deret Taylor untuk $U_{j+1,k}$:

$$u(x_{j+1}, y_k) = u(x_j, y_k) + \Delta x u_x(x_j, y_k) + \frac{\Delta x^2}{2} u_{xx}(x_j, y_k) + \frac{\Delta x^3}{6} u_{xxx}(x_j, y_k) + \frac{\Delta x^4}{24} u_{xxxx}(x_j, y_k) + \cdots \quad (2.38)$$

Ekspansi deret Taylor untuk $U_{j-1,k}$:

$$u(x_{j-1}, y_k) = u(x_j, y_k) - \Delta x u_x(x_j, y_k) + \frac{\Delta x^2}{2} u_{xx}(x_j, y_k) - \frac{\Delta x^3}{6} u_{xxx}(x_j, y_k) + \frac{\Delta x^4}{24} u_{xxxx}(x_j, y_k) - \cdots \quad (2.39)$$

Ekspansi deret Taylor untuk $U_{j,k+1}$:

$$u(x_j, y_{k+1}) = u(x_j, y_k) + \Delta y u_y(x_j, y_k) + \frac{\Delta y^2}{2} u_{yy}(x_j, y_k) + \frac{\Delta y^3}{6} u_{yyy}(x_j, y_k) + \frac{\Delta y^4}{24} u_{yyyy}(x_j, y_k) + \cdots \quad (2.40)$$

Ekspansi deret Taylor untuk $U_{j,k-1}$:

$$u(x_j,y_{k-1}) = u(x_j,y_k) - \Delta y u_y(x_j,y_k) + \frac{\Delta y^2}{2}u_{yy}(x_j,y_k) - \frac{\Delta y^3}{6}u_{yyy}(x_j,y_k) + \frac{\Delta y^4}{24}u_{yyyy}(x_j,y_k) - \cdots \quad (2.41)$$

Substitusikan persamaan (2.39) sampai (2.41) ke persamaan (2.37), sebagai berikut:

$$U_{j,k} - \frac{\Delta y^2}{2(\Delta x^2+\Delta y^2)}\Big(u(x_j,y_k) + \Delta x u_x(x_j,y_k) + \frac{\Delta x^2}{2}u_{xx}(x_j,y_k) + \frac{\Delta x^3}{6}u_{xxx}(x_j,y_k) + \frac{\Delta x^4}{24}u_{xxxx}(x_j,y_k) + \cdots + u(x_j,y_k) - \Delta x u_x(x_j,y_k) + \frac{\Delta x^2}{2}u_{xx}(x_j,y_k) - \frac{\Delta x^3}{6}u_{xxx}(x_j,y_k) + \frac{\Delta x^4}{24}u_{xxxx}(x_j,y_k) - \cdots\Big) - \frac{\Delta x^2}{2(\Delta x^2+\Delta y^2)}\Big(u(x_j,y_k) + \Delta y u_y(x_j,y_k) + \frac{\Delta y^2}{2}u_{yy}(x_j,y_k) + \frac{\Delta y^3}{6}u_{yyy}(x_j,y_k) + \frac{\Delta y^4}{24}u_{yyyy}(x_j,y_k) + \cdots + u(x_j,y_k) - \Delta y u_y(x_j,y_k) + \frac{\Delta y^2}{2}u_{yy}(x_j,y_k) - \frac{\Delta y^3}{6}u_{yyy}(x_j,y_k) + \frac{\Delta y^4}{24}u_{yyyy}(x_j,y_k) - \cdots\Big) = \frac{\Delta x^2\Delta y^2}{2(\Delta x^2+\Delta y^2)}f_{j,k}$$

Terdapat suku-suku yang dapat dikanselasi, sehingga menjadi

$$U_{j,k} - \frac{\Delta y^2}{2(\Delta x^2+\Delta y^2)}\Big(u(x_j,y_k) + \frac{\Delta x^2}{2}u_{xx}(x_j,y_k) + \frac{\Delta x^4}{24}u_{xxxx}(x_j,y_k) + \cdots + u(x_j,y_k) + \frac{\Delta x^2}{2}u_{xx}(x_j,y_k) + \frac{\Delta x^4}{24}u_{xxxx}(x_j,y_k) - \cdots\Big) - \frac{\Delta x^2}{2(\Delta x^2+\Delta y^2)}\Big(u(x_j,y_k) + \frac{\Delta y^2}{2}u_{yy}(x_j,y_k) + \frac{\Delta y^4}{24}u_{yyyy}(x_j,y_k) + \cdots + u(x_j,y_k) + \frac{\Delta y^2}{2}u_{yy}(x_j,y_k) + \frac{\Delta y^4}{24}u_{yyyy}(x_j,y_k) - \cdots\Big) = \frac{\Delta x^2\Delta y^2}{2(\Delta x^2+\Delta y^2)}f_{j,k}$$

atau

$$-\frac{\Delta x^2\Delta y^2}{2(\Delta x^2+\Delta y^2)}\big(u_{xx}(x_j,y_k) + u_{yy}(x_j,y_k)\big) - \Delta x^2\Delta y^2\left(\frac{\Delta x^2 u_{xxxx}(x_j,y_k) + \Delta y^2 u_{yyyy}(x_j,y_k)}{24(\Delta x^2+\Delta y^2)}\right) = \frac{\Delta x^2\Delta y^2}{2(\Delta x^2+\Delta y^2)}f_{j,k}$$

Setiap ruas dikalikan dengan $\frac{2(\Delta x^2+\Delta y^2)}{\Delta x^2\Delta y^2}$, sehingga

$$u_{xx}(x_j,y_k) - u_{yy}(x_j,y_k) - \frac{\Delta x^2 u_{xxxx}(x_j,y_k) + \Delta y^2 u_{yyyy}(x_j,y_k)}{12} = f_{j,k}$$

atau

$$u_{xx}(x_j,y_k) - u_{yy}(x_j,y_k) - O(\Delta x^2,\Delta y^2) = f_{j,k}$$

Jadi, kesalahan pemotongan ada pada $O(\Delta x^2,\Delta y^2)$.

3. Hasil Dan Pembahasan

Penyelesaian persamaan poisson kasus 1

$$-u_{xx} - u_{yy} = x + y$$

dengan domain

$$0 \le x \le 1$$
$$0 \le y \le 1$$

dan syarat batas
batas kiri: $u = 0$
batas kanan: $u = 0$
batas bawah: $u = 0$
batas atas: $u = 0$
dan dipilih

$$\Delta x = \Delta y = \frac{1}{4}.$$

Dengan nilai

$$f_{2,2} = f(x(2),y(2)) = \left(\frac{1}{4}+\frac{1}{4}\right) = \frac{1}{2} \qquad f_{3,2} = f(x(3),y(2)) = \left(\frac{2}{4}+\frac{1}{4}\right) = \frac{3}{4}$$

$$f_{2,3} = f(x(2),y(3)) \qquad f_{3,3} = f(x(3),y(3))$$

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$= \left(\frac{1}{4}+\frac{2}{4}\right)$ $= \left(\frac{2}{4}+\frac{2}{4}\right)$ $= \left(\frac{3}{4}+\frac{1}{4}\right)$ $= \left(\frac{3}{4}+\frac{2}{4}\right)$

$= \frac{3}{4}$ $= 1$ $= 1$ $= \frac{5}{4}$

$f_{2,4} = f(x(2), y(4))$ $f_{3,4} = f(x(3), y(4))$ $f_{4,4} = f(x(4), y(4))$

$= \left(\frac{1}{4}+\frac{3}{4}\right)$ $= \left(\frac{2}{4}+\frac{3}{4}\right)$ $= \left(\frac{3}{4}+\frac{3}{4}\right)$

$= 1$ $= \frac{5}{4}$ $= \frac{6}{4}$

$f_{4,2} = f(x(4), y(2))$ $f_{4,3} = f(x(4), y(3))$

Dari skema umum pada persamaan (2.6), dengan $\theta_x = \frac{1}{4}, \theta_y = \frac{1}{4}$ dan $\theta_{xy} = \frac{1}{64}$, untuk $J = K = 5$, pada (titik *j*, titik *k*)

$$(2,2) \Rightarrow U_{2,2} - \theta_x U_{3,2} - \theta_y U_{2,3} = \theta_{xy} f_{2,2} + \theta_x U_{1,2} + \theta_y U_{2,1}$$

$$\Rightarrow U_{2,2} - \frac{1}{4}U_{3,2} - \frac{1}{4}U_{2,3} = \frac{1}{64}.\frac{1}{2} + \frac{1}{4}.0 + \frac{1}{4}.0 \tag{3.1}$$

$$\Rightarrow U_{2,2} - \frac{1}{4}U_{3,2} - \frac{1}{4}U_{2,3} = 0{,}0078$$

$$(3,2) \Rightarrow U_{3,2} - \theta_x U_{4,2} - \theta_x U_{2,2} - \theta_y U_{3,3} = \theta_{xy} f_{3,2} + \theta_y U_{3,1}$$

$$\Rightarrow U_{3,2} - \frac{1}{4}U_{4,2} - \frac{1}{4}U_{2,2} - \frac{1}{4}U_{3,3} = \frac{1}{64}.\frac{3}{4} + \frac{1}{4}.0 \tag{3.2}$$

$$\Rightarrow U_{3,2} - \frac{1}{4}U_{4,2} - \frac{1}{4}U_{2,2} - \frac{1}{4}U_{3,3} = 0{,}0117$$

$$(4,2) \Rightarrow U_{4,2} - \theta_x U_{3,2} - \theta_y U_{4,3} = \theta_{xy} f_{4,2} + \theta_x U_{5,2} + \theta_y U_{4,1}$$

$$\Rightarrow U_{4,2} - \frac{1}{4}U_{3,2} - \frac{1}{4}U_{4,3} = \frac{1}{64}.1 + \frac{1}{4}.0 + \frac{1}{4}.0 \tag{3.3}$$

$$\Rightarrow U_{4,2} - \frac{1}{4}U_{3,2} - \frac{1}{4}U_{4,3} = 0{,}0156$$

$$(2,3) \Rightarrow U_{2,3} - \theta_x U_{3,3} - \theta_y U_{2,4} - \theta_y U_{2,2} = \theta_{xy} f_{2,3} + \theta_x U_{1,3}$$

$$\Rightarrow U_{2,3} - \frac{1}{4}U_{3,3} - \frac{1}{4}U_{2,4} - \frac{1}{4}U_{2,2} = \frac{1}{64}.\frac{3}{4} + \frac{1}{4}.0 \tag{3.4}$$

$$\Rightarrow U_{2,3} - \frac{1}{4}U_{3,3} - \frac{1}{4}U_{2,4} - \frac{1}{4}U_{2,2} = 0{,}0117$$

$$(3,3) \Rightarrow U_{3,3} - \theta_x U_{4,3} - \theta_x U_{2,3} - \theta_y U_{3,4} - \theta_y U_{3,2} = \theta_{xy} f_{3,3}$$

$$\Rightarrow U_{3,3} - \frac{1}{4}U_{4,3} - \frac{1}{4}U_{2,3} - \frac{1}{4}U_{3,4} - \frac{1}{4}U_{3,2} = \frac{1}{64}.1 \tag{3.5}$$

$$\Rightarrow U_{3,3} - \frac{1}{4}U_{4,3} - \frac{1}{4}U_{2,3} - \frac{1}{4}U_{3,4} - \frac{1}{4}U_{3,2} = 0{,}0156$$

$$(4,3) \Rightarrow U_{4,3} - \theta_x U_{3,3} - \theta_y U_{4,4} - \theta_y U_{4,2} = \theta_{xy} f_{4,3} + \theta_x U_{5,3}$$

$$\Rightarrow U_{4,3} - \frac{1}{4}U_{3,3} - \frac{1}{4}U_{4,4} - \frac{1}{4}U_{4,2} = \frac{1}{64}.\frac{5}{4} + \frac{1}{4}.0 \tag{3.6}$$

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$$\Rightarrow U_{4,3} - \frac{1}{4}U_{3,3} - \frac{1}{4}U_{4,4} - \frac{1}{4}U_{4,2} = 0{,}0195$$

$$(2{,}4) \Rightarrow U_{2,4} - \theta_x U_{3,4} - \theta_y U_{2,3} = \theta_{xy} f_{2,4} + \theta_x U_{1,4} + \theta_y U_{2,5}$$

$$\Rightarrow U_{2,4} - \frac{1}{4}U_{3,4} - \frac{1}{4}U_{2,3} = \frac{1}{64}.1 + \frac{1}{4}.0 + \frac{1}{4}.0 \tag{3.7}$$

$$\Rightarrow U_{2,4} - \frac{1}{4}U_{3,4} - \frac{1}{4}U_{2,3} = 0{,}0156$$

$$(3{,}4) \Rightarrow U_{3,4} - \theta_x U_{4,4} - \theta_x U_{2,4} - \theta_y U_{3,3} = \theta_{xy} f_{3,4} + \theta_y U_{3,5}$$

$$\Rightarrow U_{3,4} - \frac{1}{4}U_{4,4} - \frac{1}{4}U_{2,4} - \frac{1}{4}U_{3,3} = \frac{1}{64}.\frac{5}{4} + \frac{1}{4}.0 \tag{3.8}$$

$$\Rightarrow U_{3,4} - \frac{1}{4}U_{4,4} - \frac{1}{4}U_{2,4} - \frac{1}{4}U_{3,3} = 0{,}0195$$

$$(4{,}4) \Rightarrow U_{4,4} - \theta_x U_{3,4} - \theta_y U_{4,3} = \theta_{xy} f_{4,4} + \theta_x U_{5,4} + \theta_y U_{4,5}$$

$$\Rightarrow U_{4,4} - \frac{1}{4}U_{3,4} - \frac{1}{4}U_{4,3} = \frac{1}{64}.\frac{6}{4} + \frac{1}{4}.0 + \frac{1}{4}.0 \tag{3.9}$$

$$\Rightarrow U_{4,4} - \frac{1}{4}U_{3,4} - \frac{1}{4}U_{4,3} = 0{,}0234$$

Dari persamaan (3.1) sampai dengan (3.9) diatas dapat dibentuk suatu sistem persamaan linier sebagai berikut:

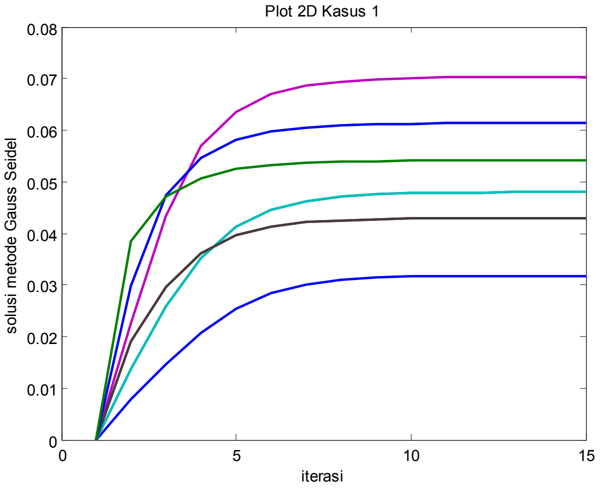
$$\mathbf{A}\mathbf{U} = \mathrm{b}$$

$$\begin{bmatrix} 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 1 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} U_{2,2} \\ U_{3,2} \\ U_{4,2} \\ U_{2,3} \\ U_{3,3} \\ U_{4,3} \\ U_{2,4} \\ U_{3,4} \\ U_{4,4} \end{bmatrix} = \begin{bmatrix} 0{,}0078 \\ 0{,}0117 \\ 0{,}0156 \\ 0{,}0117 \\ 0{,}0156 \\ 0{,}0195 \\ 0{,}0156 \\ 0{,}0195 \\ 0{,}0234 \end{bmatrix}$$

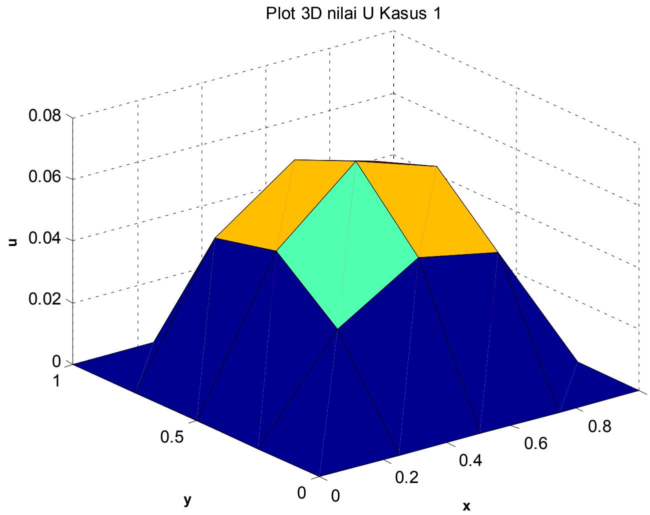
Dengan menggunakan metode Gauss-Seidel dan Conjugate-Gradient, didapatkan nilai $\mathbf{U}$ sebagai berikut:

$$\begin{bmatrix} U_{2,2} \\ U_{3,2} \\ U_{4,2} \\ U_{2,3} \\ U_{3,3} \\ U_{4,3} \\ U_{2,4} \\ U_{3,4} \\ U_{4,4} \end{bmatrix} = \begin{bmatrix} 0{,}0318 \\ 0{,}0480 \\ 0{,}0430 \\ 0{,}0480 \\ 0{,}0703 \\ 0{,}0614 \\ 0{,}0430 \\ 0{,}0614 \\ 0{,}0541 \end{bmatrix}.$$

Hasil plot solusi dengan menggunakan metode Gauss-Seidel terlihat pada Gambar 2 dan Gambar 3 sebagai berikut:

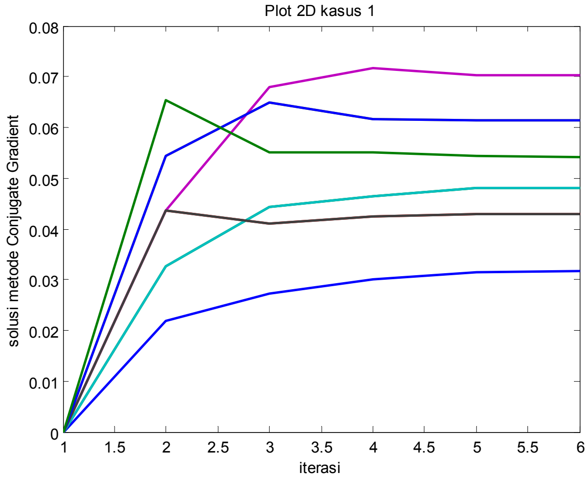


Gambar 2 Plot 2D solusi metode Gauss-Seidel

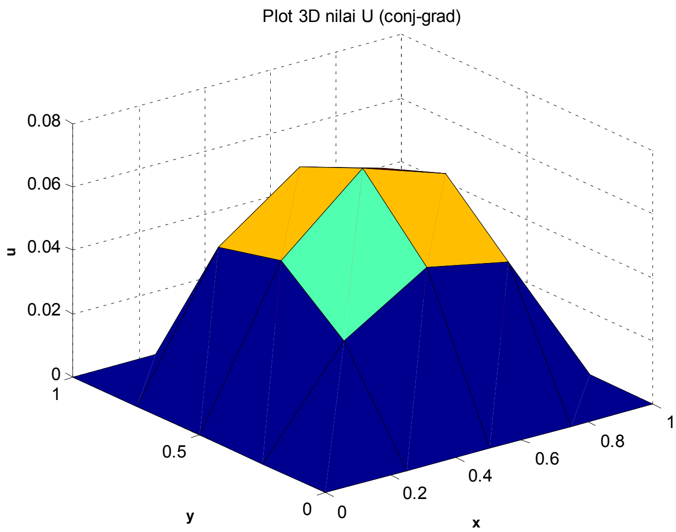


Gambar 3 Plot 3D solusi metode Gauss-Seidel

Hasil plot solusi dengan menggunakan metode Conjugate Gradient terlihat pada Gambar 4 dan Gambar 5 sebagai berikut:



Gambar 4 Plot 2D solusi metode Conjugate Gradient



Gambar 5 Plot 3D solusi metode Conjugate Gradient

4. KESIMPULAN

Dari hasil plot menggunakan Software MATLAB seperti terlihat pada Gambar 2, Gambar 3, Gambar 4, dan Gambar 5 dapat diketahui bahwa nilai $\mathbf{U}$ yang dihasilkan metode Gauss-Seidel dan Conjugate Gradient adalah sama. Namun, pada proses iterasinya, dengan menggunakan metode Conjugate Gradient nilai konvergen setelah iterasi ke-5. Sedangkan dengan metode Gauss-Seidel nilai konvergen setelah iterasi ke-14.

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